EXERCISES FOR ELGA3 MINICOURSE: BIRATIONAL GEOMETRY OF MODULI SPACES OF SHEAVES

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The survey article "Birational geometry of moduli spaces of sheaves and Bridgeland stability," as well as your lecture notes, should contain the definitions and background necessary for most of the following exercises.

Problem 1. Let X be a smooth projective curve over $\mathbb{C}$. Show that the Hilbert scheme $X^{[n]}$ and the symmetric product $X^{(n)}$ are isomorphic. (Hint: classify length n zero-dimensional subschemes of X using the fact that the local rings $\mathcal{O}_{X,p}$ are DVRs.)

Problem 2. Let X be a smooth projective variety. Recall that Serre's criterion for ampleness says that a line bundle L on X is ample if for every coherent sheaf F on X there exists some $n \gg 0$ such that $F \otimes L^{\otimes n}$ is globally generated.

1. Show that if L and M are ample line bundles on X then $L \otimes M$ is ample.
2. Suppose L is ample and M is an arbitrary line bundle. Show that if $n \gg 0$ then $L^{\otimes n} \otimes N$ is ample.

Deduce that the cone $\mathrm{Amp}(X) \subset N^1(X)$ spanned by the classes of ample Cartier divisors is open, and that a $\mathbb{Q}$-Cartier divisor D is ample if and only if D is in the ample cone. (Recall that ampleness is a numerical property by the Nakai-Moishezon criterion.)

Problem 3. Let $\mathbb{F}_e$ ($e \geq 0$) be the Hirzebruch surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(e) \oplus \mathcal{O}_{\mathbb{P}^1})$, which is a ruled surface over $\mathbb{P}^1$ with a section E of self-intersection $-e$. If F is the class of a fiber, then $\mathrm{Pic}(\mathbb{F}_e) \cong \mathbb{Z}F \oplus \mathbb{Z}E$, so $N^1(X)$ is a two-dimensional vector space. (Hartshorne V.2 or Beauville "Complex Algebraic Surfaces" Chapter 4 can be useful for background.)

1. Determine the ample (equivalently, nef) and effective cones of divisors on $\mathbb{F}_e$.
2. Decompose the effective cone into chambers according to stable base locus, and determine the rational maps corresponding to the chambers.

Problem 4. Let $X = \mathrm{Bl}_{p,q,r}\,\mathbb{P}^2$ be the blowup of $\mathbb{P}^2$ at three points p, q, r.

1. Determine the Picard group of X.
2. Suppose p, q, r are not collinear. Compute the nef and effective cones of divisors on X. (Hint: the effective cone is spanned by the 6 curves E_i and $H - E_i - E_j$ given by exceptional divisors and lines passing between two of the points.)
3. What maps do the extremal rays of the nef cone in (2) correspond to?
4. Repeat (2) when p, q, r are collinear.

Problem 5. Determine the effective cone of $\mathbb{P}^{2[4]}$ and compute its stable base locus decomposition. (See §10 of Arcara, Bertram, Coskun, Huizenga "The minimal model program for the Hilbert scheme of points on $\mathbb{P}^2$ and Bridgeland stability" for a full solution and the next several cases.)

Problem 6. Here we compute the effective cone of $\mathbb{P}^{2[6]}$.

1. Review for yourself why B is an extremal effective divisor.

Date: August 4, 2017.

(2) Use test curves to prove directly that the locus

$$\{Z : Z \text{ lies on a conic}\} \subset \mathbb{P}^{2[6]}$$

is a divisor of class $2H^{[6]} - \frac{1}{2}B$.

(3) Show that 6 general points in $\mathbb{P}^2$ lie on a smooth cubic $C \subset \mathbb{P}^2$, and that these points move in a pencil on C to give a $6:1$ map $C \to \mathbb{P}^1$ (use Riemann-Roch). This map induces a curve $C_{[6]}$ on $\mathbb{P}^{2[6]}$. Compute the intersection numbers $C_{[6]} \cdot H^{[6]}$ and $C_{[6]} \cdot B$ (using Riemann-Hurwitz for the latter). Conclude that $C_{[6]}$ is a moving curve class orthogonal to the divisor $2H^{[6]} - \frac{1}{2}B$. Therefore this is an extremal effective divisor.

Problem 7. Suppose $Z \subset \mathbb{P}^2$ is a collection of n distinct points. Show that the subspace

$$H^0(I_Z(n-1)) \subset H^0(\mathcal{O}_{\mathbb{P}^2}(n-1))$$

has codimension n. (Hint: consider the exact sequence

$$0 \to I_Z(n-1) \to \mathcal{O}_{\mathbb{P}^2}(n-1) \to \mathcal{O}_Z(n-1) \to 0$$

and show that the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^2}(n-1)) \to H^0(\mathcal{O}_Z(n-1)) = \mathbb{C}^n$$

is surjective by showing that the standard basis elements of $\mathbb{C}^n$ are in the image.)

Problem 8. Here we compute the effective cone of $\mathbb{P}^{2[12]}$.

(1) Let $T_{\mathbb{P}^2}$ be the tangent bundle. Use the Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 3} \to T_{\mathbb{P}^2} \to 0$$

to show that $h^0(T_{\mathbb{P}^2}(2)) = 24$.

(2) The ideal sheaf I_Z of a general collection of n points in $\mathbb{P}^2$ has a *Gaeta resolution*. In the case $n = 12$, this takes the form

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-6)^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^2}(-4)^{\oplus 3} \to I_Z \to 0.$$

Use this to show that for a general collection Z of 12 points, we have $H^0(T_{\mathbb{P}^2}(2) \otimes I_Z) = 0$.

(3) Compute the Chern classes of $T_{\mathbb{P}^2}(2)$ using the Whitney sum formula to conclude that the locus

$$D_{T_{\mathbb{P}^2}(2)} = \{Z : H^0(T_{\mathbb{P}^2}(2) \otimes I_Z) \neq 0\}$$

is an effective divisor of class

$$[D_{T_{\mathbb{P}^2}(2)}(2)] = 7H^{[12]} - B.$$

(4) Show that the curve on $\mathbb{P}^{2[12]}$ given by letting 12 points move in a pencil on a smooth plane quartic is a moving curve class that intersects the above divisor in 0. Conclude that the divisor is an extremal effective divisor.

Problem 9. Let $0 \neq s \in \operatorname{Ext}^1(\mathcal{O}_{\mathbb{P}^1}(1), \mathcal{O}_{\mathbb{P}^1}(-1)) \cong \mathbb{C}$ be a nontrivial extension class, and let E_s be the corresponding bundle

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-1) \to E_s \to \mathcal{O}_{\mathbb{P}^1}(1) \to 0.$$

Show that $E_s \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$. (You may want to use Grothendieck's theorem that any vector bundle on $\mathbb{P}^1$ decomposes as a direct sum of line bundles. If you haven't learned this, it is a good theorem to look up.)

Problem 10. Let X be a smooth curve, and let E be a torsion-free sheaf on X. Show that E is locally free (i.e. a vector bundle).

Problem 11. Let X be a smooth curve and let E be a vector bundle on X. Use the Riemann-Roch formula for vector bundles on a curve

$$\chi(E) = c_1(E) + r(E)(1 - g(X))$$

to show that E is semistable (in the reduced Hilbert polynomial sense) if and only if for every nonzero sub-bundle $F \subset E$ we have $\mu(F) \leq \mu(E)$.

Problem 12. Let X be a smooth projective variety and suppose

$$0 \to F \to E \to Q \to 0$$

is an exact sequence of sheaves which are all pure of the same dimension d. (Possibly d is smaller than $\dim X$, so that the sheaves can be torsion.) Prove the see-saw property: the reduced Hilbert polynomials satisfy $p_F \leq p_E$ if and only if $p_E \leq p_Q$.

Problem 13. Let X be a smooth projective variety and let F, E be stable sheaves on X.

(1) Show that if the reduced Hilbert polynomials satisfy $p_F = p_E$ then any nonzero homomorphism $F \to E$ is an isomorphism.
(2) Show that the only homomorphisms $E \to E$ are scalar multiples of the identity. (We're over $\mathbb{C}$!)

Problem 14. Let X be a smooth surface, and let $Z, Z' \in X^{[n]}$ be two subschemes of length n. Show that the structure sheaves $\mathcal{O}_Z$ and $\mathcal{O}_{Z'}$ are S-equivalent if and only if Z and Z' map to the same point in the symmetric product $X^{(n)}$ under the Hilbert-Chow morphism. Therefore, the moduli space $M(\operatorname{ch} \mathcal{O}_Z)$ is isomorphic to the symmetric product $X^{(n)}$.

Explain how the moduli property of the moduli space $M(\operatorname{ch} \mathcal{O}_Z)$ and the universal family on $X^{[n]}$ can be used to define the Hilbert-Chow morphism $h : X^{[n]} \to X^{(n)}$.

Problem 15. Recall that for $\beta \in \mathbb{R}$ we defined categories

$$\begin{aligned}
\mathcal{T}_\beta &= \{E \in \operatorname{Coh}(\mathbb{P}^2) : \text{every quotient } E \to Q \text{ has } \mu(Q) > \beta\} \\
\mathcal{F}_\beta &= \{E \in \operatorname{Coh}(\mathbb{P}^2) : \text{every subsheaf } F \subset E \text{ has } \mu(F) \leq \beta\} \\
\mathcal{A}_\beta &= \{E^\bullet \in D^b(\operatorname{Coh}(\mathbb{P}^2)) : \mathrm{H}^0(E^\bullet) \in \mathcal{T}_\beta,\ \mathrm{H}^{-1}(E^\bullet) \in \mathcal{F}_\beta,\ \mathrm{H}^i(E^\bullet) = 0 \text{ for } i \neq 0, 1\}
\end{aligned}$$

Let $E \in \mathcal{T}_\beta \subset \mathcal{A}_\beta$ be a sheaf, let $F \in \mathcal{A}_\beta$, and let $\phi : F \to E$ be any map. Show that ϕ is an injection in the (abelian) category $\mathcal{A}_\beta$ if and only if $F \in \mathcal{T}_\beta$ and in the natural exact sequence of sheaves

$$0 \to K \to F \xrightarrow{\phi} E \to C \to 0$$

we have $K \in \mathcal{F}_\beta$ and $C \in \mathcal{T}_\beta$. (Read Gelfand-Manin, "Methods in Homological Algebra" for the necessary background.)

Problem 16. The stability condition $\sigma_{\beta,\alpha} = (Z_{\beta,\alpha}, \mathcal{A}_\beta)$ on $\mathbb{P}^2$ has central charge defined by the formula

$$Z_{\beta,\alpha}(E) = -\int_{\mathbb{P}^2} e^{-(\alpha+i\beta)H} \cdot \operatorname{ch}(E),$$

where $\operatorname{ch}(E)$ is the Chern character

$$\operatorname{ch}(E) = \operatorname{ch}_0(E) + \operatorname{ch}_1(E) + \operatorname{ch}_2(E) = r(E) + c_1(E) + \frac{1}{2}(c_1(E)^2 - 2c_2(E)).$$

Expand this to get an explicit formula for $Z_{\beta,\alpha}(E)$ in terms of the the Chern characters $\operatorname{ch}_i(E)$.

Problem 17. If $\mathbf{v} \in K(\mathbb{P}^2)$ is a set of Chern classes with positive rank $r(\mathbf{v}) > 0$, we defined the slope and discriminant by

$$\mu(\mathbf{v}) = \frac{c_1(\mathbf{v})}{r(\mathbf{v})}$$
$$\Delta(\mathbf{v}) = \frac{1}{2}\mu(\mathbf{v})^2 - \frac{\mathrm{ch}_2(\mathbf{v})}{r(\mathbf{v})}.$$

Show that in this case the Bridgeland slope becomes

$$\mu_{\sigma_{\beta,\alpha}}(\mathbf{v}) = \frac{(\mu(\mathbf{v}) - \beta)^2 - \alpha^2 - 2\Delta(\mathbf{v})}{\mu(\mathbf{v}) - \beta}$$

Problem 18. Suppose $\mathbf{v}, \mathbf{w} \in K(\mathbb{P}^2)$ have positive rank and different slope. Show that the wall

$$W(\mathbf{v}, \mathbf{w}) := \{(\beta, \alpha) : \mu_{\sigma_{\beta,\alpha}}(\mathbf{v}) = \mu_{\sigma_{\beta,\alpha}}(\mathbf{w})\}$$

is the semicircle in the (β, α)-half plane with center $(s_W, 0)$ and radius ρ_W satisfying

$$s_W = \frac{\mu(\mathbf{v}) + \mu(\mathbf{w})}{2} - \frac{\Delta(\mathbf{v}) - \Delta(\mathbf{w})}{\mu(\mathbf{v}) - \mu(\mathbf{w})}$$
$$\rho_W^2 = (s_W - \mu(\mathbf{v}))^2 - 2\Delta(\mathbf{v}).$$

Problem 19. Determine all the actual walls in the (β, α)-half plane for the Chern character $\mathbf{v} = \mathrm{ch}\, I_Z$, where $Z \in \mathbb{P}^{2[3]}$. (Hint: either 3 points lie on a line, and so I_Z fits as an extension

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-1) \to I_Z \to \mathcal{O}_L(-3) \to 0,$$

or they do not lie on a line and I_Z has minimal free resolution

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-3)^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus 3} \to I_Z \to 0.$$

We can view this as exhibiting I_Z as an extension

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus 3} \to I_Z \to \mathcal{O}_{\mathbb{P}^2}(-3)^{\oplus 2}[1] \to 0$$

in any of the categories $\mathcal{A}_\beta$ with $-3 < \beta < -2$. Show that there are precisely two walls, corresponding to each of these geometric possibilities. You may want to use and/or look up the following fact (see Prop 6.5 in the survey for a reference): line bundles $\mathcal{O}_{\mathbb{P}^2}(a)$ are $\sigma_{\beta,\alpha}$-stable objects of $\mathcal{A}_\beta$ whenever $\beta < a$.)

Compare your picture in the Bridgeland plane with the stable base locus decomposition of $\mathbb{P}^{2[3]}$, worked out in class.