A bit(e) of toric varieties

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1. Projective space: the torus, the orbits, the invariant divisors. Definition of a toric variety
2. Affine toric varieties from cones and affine semigroups. Toric varieties from fans
3. Projective toric varieties. Toric varieties from polytopes
4. Divisors and line bundles
5. Toric varieties as quotients

The torus

For simplicity we will work over the field $k = \mathbb{C}$.

What is a torus

- A torus is an algebraic group isomorphic to $(\mathbb{C}^*)^n$ with componentwise multiplication:

$$x \cdot y = (x_1 y_1, \ldots, x_n y_n).$$

- $T = (\mathbb{C}^*)^n$: the associated ring is the semigroup algebra $\mathbb{C}[\mathbb{Z}^n]$, isomorphic to

$$\mathbb{C}[x_1, \ldots, x_n, y]/\langle x_1 \ldots x_n y - 1\rangle \simeq$$
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the ring of Laurent polynomials $f = \sum_{\text{finite subset of } \mathbb{Z}^n} c_m x^m$, $c_m \in \mathbb{C}$.

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Projective space

The torus $T \subset \mathbb{P}^n$

- $\mathbb{P}^n = (\mathbb{C})^{n+1} \setminus \{0\}/\equiv$, with homogeneous coordinates $(x_0 : \cdots : x_n)$.
- $T(\mathbb{P}^n) = (\mathbb{C}^*)^{n+1} \setminus \{0\}/\equiv$, with coordinatewise multiplication.
- Two points $x, y \in \mathbb{P}^n$ are in the same orbit if and only if they have the same *support*.
- The torus $T(\mathbb{P}^n)$ is the orbit of the point $(1 : \cdots : 1)$.
- There are $n+1$ fixed points $(1 : 0 : \cdots : 0), \ldots, (0 : \cdots : 0 : 1)$ and $n+1$ invariant hypersurfaces $(x_i = 0), i = 0, \ldots, n$.
- $\mathbb{P}^n$ is equal to the union of the $(n+1)$ invariant open sets $(x_i \neq 0), i = 1, \ldots, n$.

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Definition

Toric varieties

A *toric variety* is a (normal) algebraic variety X (over $\mathbb{C}$) with a dense Zariski open torus $T \subseteq X$, such that the torus group operation extends to an algebraic action on X. That is, there is a morphism $T \times X \to X$ which makes the following diagram commutative.

$$\begin{array}{ccc} T \times T & \longrightarrow & T \\ {\scriptstyle 1\times i}\downarrow & & \downarrow{\scriptstyle i} \\ T \times X & \longrightarrow & X \end{array}$$

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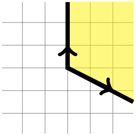
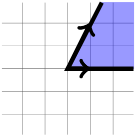
Cones and their duals

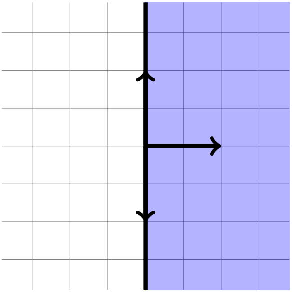
- N, M dual (fin. gen.) lattices, $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$.
- σ a (rational) cone in $N_{\mathbb{R}}$, the dual (rational) cone (both polyhedral) $\sigma^\vee =$

  $$\{m \in M_{\mathbb{R}} : \langle m, x\rangle \geq 0,\ x \in \sigma\}.$$

- $\sigma = Cone(e_2, 2e_1 - e_2)$, $\sigma^\vee = Cone(e_1, e_1 + 2e_2)$:

- $\sigma = Cone(e_1)$, $\sigma^\vee = Cone(e_1, e_2, -e_2)$:



σ is strictly convex if and only if $\sigma^\vee$ is full dimensional.

Affine toric varieties

- Given $N \simeq \mathbb{Z}^n$, we define the torus $T_N = N \otimes_{\mathbb{Z}} \mathbb{C}^* \simeq (\mathbb{C}^*)^n$. Its coordinate ring is the semigroup algebra $\mathbb{C}[M] \simeq \mathbb{C}[\mathbb{Z}^n]$.
- Given any rational polyhedral cone σ in $N_{\mathbb{R}}$, we define the semigroup ring $S_\sigma = \sigma^\vee \bigcap M$, which is finitely generated by Gordan's lemma. We then define the variety

$$X_\sigma = Spec(S_\sigma).$$

- Then
  1. X_σ is a (normal) toric variety with torus T_N.
  2. All affine normal toric varieties are of this form.

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Affine toric varieties

- We have: $N \simeq \mathbb{Z}^n$, $M = Hom_{\mathbb{Z}}(N, \mathbb{Z}) = N^\nu$, $T_N = N \otimes_{\mathbb{Z}} \mathbb{C}^* \simeq (\mathbb{C}^*)^n$, $\mathbb{C}[M] \simeq \mathbb{C}[\mathbb{Z}^n]$, $\sigma \subset N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^n$, $\sigma^\vee \subset M_{\mathbb{R}} = M \otimes_Z \mathbb{R} \simeq (\mathbb{R}^n)^\nu$, $S_\sigma = \sigma^\vee \bigcap M \subset M$, $X_\sigma = Spec(\mathbb{C}[S_\sigma])$.

- X_σ is a (normal) toric variety with torus T_N with coordinate ring $\mathbb{C}[S_\sigma]$:

$$S_\sigma \subset M \to \mathbb{C}[S_\sigma] \subset \mathbb{C}[M],$$

so $T_N \subset X_\sigma$.

- $\mathbb{C}$-points of X_σ are identified with semigroup morphisms $S_\sigma \to \mathbb{C}$ (the operation in $\mathbb{C}$ is the product) and $\mathbb{C}$-points of T_N are identified with semigroup morphisms $S_\sigma \to \mathbb{C}^*$.

- There is a distinguished point x_σ in X_σ, defined by the morphism of semigroups $0 \mapsto 1, m \neq 0 \mapsto 0$.

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Affine toric varieties

- σ is smooth if we can choose generators of σ that extend to a basis of N. It holds that X_σ is smooth if and only if σ is smooth.
- If $\sigma = \mathbb{R}_{\geq 0}e_1 + \ldots \mathbb{R}_{\geq 0}e_n$ is the first orthant in $\mathbb{R}^n$, $\sigma^\vee$ is the first orthant in $(\mathbb{R}^n)^\nu$ and then $X_\sigma = \mathbb{C}^n$.
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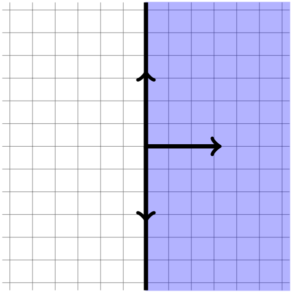

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- σ is simplicial if it can be generated by linearly independent vectors (so, at most as many as the rank n of N). It holds that X_σ is an orbifold if and only if σ is simplicial.
- $\sigma = Cone(e_2, 2e_1 - e_2)$ is simplicial but not smooth because $|\det(e_2, 2e_1 - e_2)| = 2$. Then $\sigma^\vee = \mathbb{R}_{\geq 0}$ and $S_\sigma = \langle e_1, e_1 + e_2, e_1 + 2e_2 \rangle_{\mathbb{N}}$.

- $$\mathbb{C}[S_\sigma] \simeq \mathbb{C}[x, xy, xy^2] \simeq \mathbb{C}[x, y, z]/\langle y^2 - xz \rangle.$$

  Thus, $X_\sigma \simeq V(y^2 - xz) \subseteq \mathbb{C}^3$, which is singular at $\{(0,0,0)\}$.
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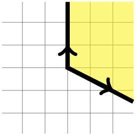
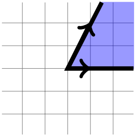

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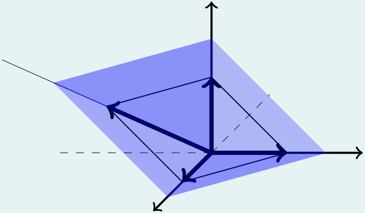

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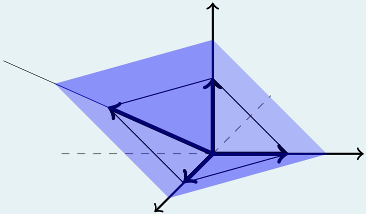

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Rational polyhedral fans and toric varieties

A rational polyhedral (r. p.) fan Σ is a finite collection of r.p. (strictly convex) cones in $N_{\mathbb{R}}$ which intersect along a common face and which contains all faces of the cones in the collection.

We can construct a toric variety X_Σ associated to such a fan, by gluing the affine toric varieties X_σ for all $\sigma \in \Sigma$, which give a cover by invariant open sets.

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Projective space arising from a fan

- Let $\Sigma \subseteq \mathbb{R}^2$ be the complete fan with rays

  $\Sigma(1) = \{e_1, e_2, e_0 = -e_1 - e_2\}.$

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- X_{σ_i} coincide with the usual cover $U_i = \{x_i \neq 0\}$ of $\mathbb{P}^2$ thought with global homogeneous coordinates $(x_0 : x_1 : x_2)$. And we get the usual glueing:

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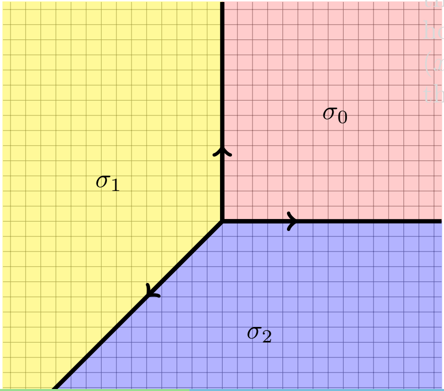
- 

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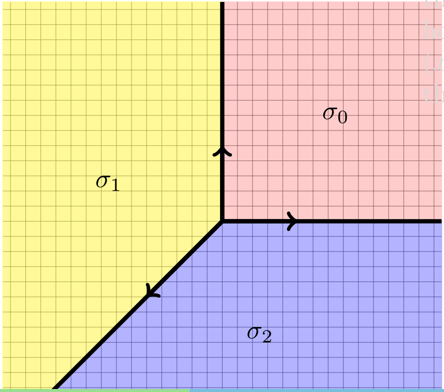
- 

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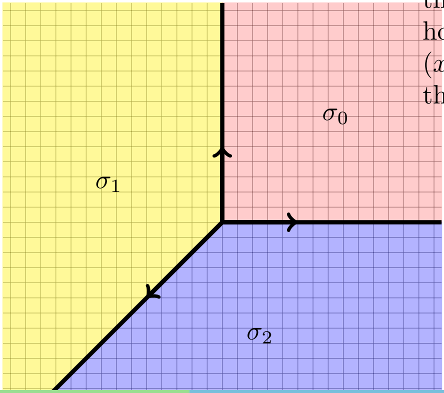
- 

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Properties of X_Σ read in Σ

Theorem

1. X_Σ is smooth if and only if each maximal cone in Σ is smooth.
2. X_Σ is an orbifold si if and only if each maximal cone in Σ is simplicial.
3. X_Σ is compact (proper) if and only if the union of the cones in Σ equals $N_\mathbb{R}$ (but not necessarily projective).

Hirzebruch surfaces

- Hirzebruch surfaces $\mathbb{H}_r, r \geq 0$, are ruled toric varieties. $\mathbb{H}_r$ is the vector bundle of $\mathbb{P}^1$ associated to the sheaf $\mathcal{O}(0) \bigoplus \mathcal{O}(-r)$. $\mathbb{H}_r = X_{\Sigma_r}$, where Σ_r is the fan:
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- When $r = 0$, we get $\mathbb{P}^1 \times \mathbb{P}^1$.
- When $r = 1$, $\mathbb{H}_1 \simeq Bl_0(\mathbb{P}^2)$
- $\mathbb{H}_r$ is smooth for any r.
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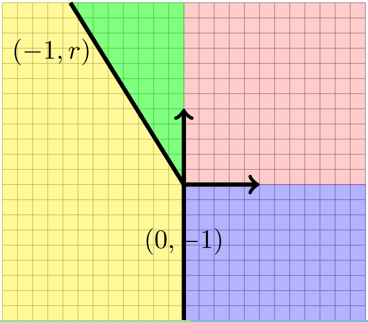
- 

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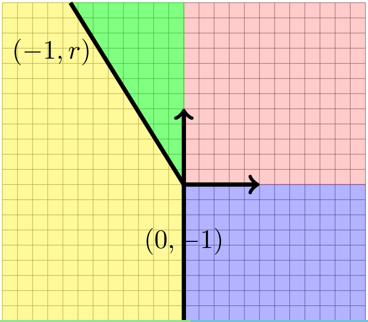
- 

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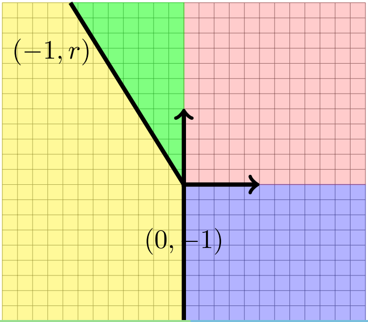
- 

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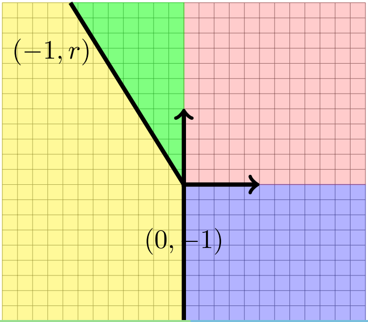
- 

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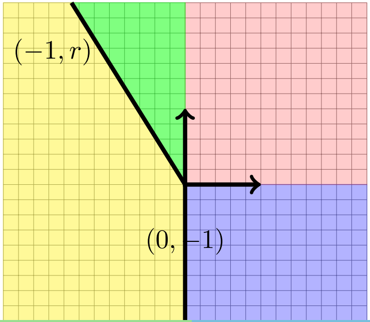
- 

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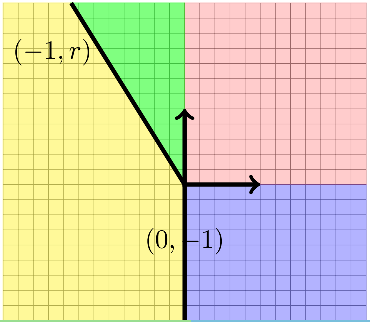
- 

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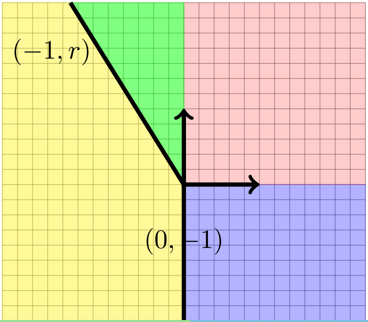
- 

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Some facts

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- Moreover, if $u \in Relint(\sigma)$, then the limit equals the distinguished point $x_\sigma \in X_\sigma$.
- This allows to recover the fan from the toric variety.
- Toric varieties are (partial) algebraic compactifications of T_N. If $\rho_1, \ldots, \rho_r$ are the rays in a fan Σ, then

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- Let $\sigma \subseteq N_{\mathbb{R}}$ and $u \in N$. Then $\lim_{t\to 0}(t^{u_1}, \ldots, t^{u_n})$ exists in X_σ if and only if $u \in \sigma$.
- Moreover, if $u \in Relint(\sigma)$, then the limit equals the distinguished point $x_\sigma \in X_\sigma$.
- This allows to recover the fan from the toric variety.
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Lattice polytopes and projective toric varieties

- A lattice polytope P is the convex hull of finitely many lattice points (i.e. integer points) in $\mathbb{R}^n$. Equivalently, a lattice polytope is compact and it is the intersection of finitely many rational halfspaces $\{x \in \mathbb{R}^n : \langle \eta, x \rangle \geq -a_\eta\}$, with η, a_η rational. In fact, we can assume that $\eta \in \mathbb{Z}^n$ is primitive and $a_\eta \in \mathbb{Z}$.
- The intersection of the hyperplane $\{x \in \mathbb{R}^n : \langle \eta, x \rangle = -a_\eta\}$ with P is a face of P, denoted F_η.
- Given a lattice polytope P, there is an associated r. p. fan Σ_P (the normal fan of P), where the cones are in bijection with the faces of P of complementary dimension:

$$F \text{ face of } P \leftrightarrow \sigma_F = closure(\{\eta : F = F_\eta\}).$$

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Divisor of a character

- $div : \mathbb{C}(X_\Sigma)^* \to Div(X_\Sigma)$ and we have the inclusion

$$\mathbb{C}[T_N] = \mathbb{C}[M] \hookrightarrow \mathbb{C}(X_\Sigma)^*.$$

- Gven $m \in M$, the character χ^m has neither zeros nor poles in T_N, and so $div(\chi^m) \in Div(X_\Sigma)$ is supported in $X_\Sigma \setminus T_N$.
- As $X_\Sigma \setminus T_N = \bigcup_{\rho \in \Sigma(1)} D_\rho$, $div(\chi^m) = \sum_\rho v_{D_\rho}(\chi^m) D_\rho$, where v_D it the valuation associated to the ring $\mathcal{O}_{X_\Sigma, D}$ for each prime divisor D.
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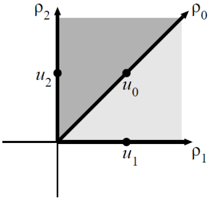
Blow-up



$u_1 = (1,0), u_2 = (0,1), u_0 = (1,1)$

Then $D(\chi^{u_1}) = \langle u_1, u_0\rangle D_0 + \langle u_1, u_1\rangle D_1 + \langle u_1, u_2\rangle D_2$,

$D(\chi^{u_2}) = D_2 + D_0$, and

$Cl(Bl_0(\mathbb{C}^2)) \simeq \mathbb{Z}$

with generator

$$[D_1] = [D_2] = -[D_0].$$

The standard map from $Bl_0(\mathbb{C}^2)$ to $\mathbb{C}^2$ arises naturally (given by the identity).

The class group of a toric variety

- The action of T_N on X_Σ induces an action on $Div(X_\Sigma)$: $t \cdot \sum a_D D \mapsto \sum a_D tD$, where tD is the image of D by multiplication by t.
- Denote by $Div_{T_N}(X_\Sigma)$ the subgroup of invariant Weil divisors.
- $Div_{T_N}(X_\Sigma) = \{\sum_{\rho \in \Sigma(1)} a_\rho D_\rho\}$.
- Given Σ, we have an exact sequence

$$M \longrightarrow Div_{T_N}(X_\Sigma) \longrightarrow Cl(X_\Sigma) \longrightarrow 0.$$

  where the first morphism is $m \mapsto div(\chi^m)$ and the second is projection to the quotient.
- Moreover the rays in $\Sigma(1)$ span $\mathbb{R}^n$ (equiv. X_Σ does not have toric factors) if and only if the first morphism is injective. In this case, we get the exact sequence:

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Examples

In the case of $\mathbb{P}^n$, there are $n+1$ rays generated by $e_0 = -\sum_i e_i, e_1, \dots, e_n$. The sequence:

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reduces to

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow \mathbb{Z}^{n+1} \longrightarrow \mathbb{Z} \longrightarrow 0,$$

where the maps equal

$$m \mapsto (-\sum_i m_i, m_1, \dots, m_n),$$

$$(a_0, \dots, a_n) \mapsto a_0 + \cdots + a_n.$$

This comes from the fact that $e_0 + e_1 + \dots e_n = 0$ and this is essentially the single (primitive) linear relation among the primitive generators of the rays.

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Examples

- In general $Cl(X_\Sigma)$ might have torsion.
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Cartier divisors

- Denote by $CDiv_{T_N}(X_\Sigma)$ the Cartier invariant divisors on X_Σ.
- As $div(M) \subseteq CDiv_{T_N}(X_\Sigma)$, we can restrict the previous exact sequence to get another exact sequence:

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Cartier divisors as an inverse limit

- In any affine toric variety it holds that $Pic(X_\sigma) = 0$.
- An invariant divisor D is Cartier if and only if for any $\sigma \in \Sigma$ there exists $m_\sigma \in M$ such that $D|_{X_\sigma} = div(\chi^{m_\sigma})|_{X_\sigma}$.
- m_σ is unique modulo $M(\sigma) = \sigma^\perp \cap M$.
- When $\tau \leq \sigma$,
$$(D|_{X_\sigma})|_{X_\tau} = D|_{X_\tau}.$$
- The poset $(\Sigma, \leq)$ together with natural restriction morphisms $\{\phi_{\tau,\sigma}\}$ form an inverse system and
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- In any affine toric variety it holds that $Pic(X_\sigma) = 0$.
- An invariant divisor D is Cartier if and only if for any $\sigma \in \Sigma$ there exists $m_\sigma \in M$ such that $D|_{X_\sigma} = div(\chi^{m_\sigma})|_{X_\sigma}$.
- m_σ is unique modulo $M(\sigma) = \sigma^\perp \cap M$.
- When $\tau \leq \sigma$,

$$(D|_{X_\sigma})|_{X_\tau} = D|_{X_\tau}.$$

- The poset $(\Sigma, \leq)$ together with natural restriction morphisms $\{\phi_{\tau,\sigma}\}$ form an inverse system and

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Cartier divisors and support functions

- Thus, given an invariant Cartier divisor D, to any cone σ in Σ there is an associated lattice point m_σ (unique modulo $M(\sigma)$, so for maximal dimensional cones it is unique).
- The linear function ψ_σ defined by $x \to \langle m_\sigma, x\rangle$ takes the value $-a_\rho$ for any ρ ray in σ.
- This defines a Weil divisor $\sum_{\rho\in\Sigma(1)} a_\rho D_\rho$.
- We can then identify $D \in CDiv_{T_N}(X_\Sigma)$ with a piecewise linear integer function ψ_D on the support of Σ whose restriction to any cone σ is the linear function ψ_σ, called a support function.
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Global sections of line bundles

- A divisor class $D \in Pic(X_\Sigma)$ is characterized by its associated sheaf $\mathcal{O}_{X_\Sigma}(D)$, which in fact is the sheaf of sections of a line bundle $\mathcal{L}_D$. All line bundles are (isomorphic) to some $\mathcal{L}_D$.
- If $D = \sum a_\rho D_\rho$ is a Weil divisor, we define the polyhedron $P_D \subseteq M_\mathbb{R}$:

$$P_D = \{x \in M_\mathbb{R} : \langle m, u_\rho \rangle \geq -a_\rho, \qquad \forall \rho \in \Sigma(1)\}.$$

- Let D be an invariant divisor on X_Σ. Then

$$\Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)) = \bigoplus_{m \in P_D} \mathbb{C} \cdot \chi^m.$$

- Linear equivalence corresponds to a translation of P_D.
- When X_Σ is complete, P_D is a polytope (bounded) and the dimension of $\Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}(D))$ is finite. The computation of these dimensions correspond to counting lattice points in polytopes (Ehrhart theory).

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The Cartier divisor of a lattice polytope

- Given a lattice polytope P presented as the intersection of the supporting halfspaces of its facets F:

$$P = \cup_F \{\langle \eta_F, x\rangle \geq -a_F\},$$

where $-a_F$ is the minimum value of the inner products $\langle \eta_F, x\rangle$ for $x \in P$, consider the toric variety $X_{\Sigma(P)}$.

- We define the invariant Weil divisor $D_P = \sum_F a_F D_F$.
- D_P is Cartier. This is easy to see:
  1. Maximal dimensional cones in Σ_P are in bijection with the vertices v of P. Call σ_v such a cone.
  2. Then $m_{\sigma_v} = v$.
- Moreover, if $n \geq 2$, then kD_P is very ample for every $k \geq n-1$.
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Complete toric varieties and line bundles

Let $S \subset N_{\mathbb{R}}$ be convex. A function $\psi : S \to \mathbb{R}$ is convex if

$$\psi(tu + (1-t)v) \geq t\psi(u) + (1-t)\psi(v), \text{ for all } u, v \in S, t \in [0,1].$$

- A torus invariant Cartier divisor D has no basepoints, i.e., $\mathcal{O}(D)$ is generated by global sections, if and only if $m_\sigma \in P_D$ for all $\sigma \in \Sigma(n)$, if and only if ψ_D is convex, if and only if $P_D = Conv(m_\sigma : \sigma \in \Sigma(n))$.
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- There is a description of X_Σ as a categorical quotient, which is a good geometric quotient if and only if X_Σ is simplicial:

$$X_\Sigma \simeq \mathbb{C}^r \setminus Z//G$$

- Applying $Hom(-, \mathbb{C}^*)$ to the exact sequence (when $\Sigma(1)$ spans $\mathbb{R}^n$):

$$0 \to M \xrightarrow{i} \mathbb{Z}^{\Sigma(1)} \xrightarrow{q} Cl(X_\Sigma) \to 0,$$

we get the exact sequence:

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- Then, $1 \to G \to (\mathbb{C}^*)^{\Sigma(1)} \xrightarrow{i_*} T_N \to 1$, and we get the isomorphism of algebraic groups:

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The variety Z

- The combinatorics of the cones enters in the definition of Z.
- π is a quotient map extending i^* :

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G & \longrightarrow & (\mathbb{C}^*)^{\Sigma(1)} & \xrightarrow{i^*} & T_N & \longrightarrow & 1 \\ & & \Big| & & \big\downarrow & & \big\downarrow & & \\ & & G & \longrightarrow & \mathbb{C}^{\Sigma(1)} \setminus Z & \xrightarrow[\pi]{} & X_\Sigma & & \end{array}$$

- The coordinate ring of $\mathbb{C}^{\Sigma(1)}$ is $\mathbb{C}[x_\rho | \rho \in \Sigma(1)]$. It has a multigrading with values in the abelian group $Cl(X_\Sigma)$.
- For each $\sigma \in \Sigma$ we define the monomial: $x^\sigma = \prod_{\rho \notin \sigma(1)} x_\rho$.
- Call C the monomial ideal $C = \langle x^\sigma \ : \ \sigma \in \Sigma_{max} \rangle$.
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- For each $\sigma \in \Sigma$ we define the monomial: $x^\sigma = \prod_{\rho \notin \sigma(1)} x_\rho$.
- Call C the monomial ideal $C = \langle x^\sigma \ : \ \sigma \in \Sigma_{max} \rangle$.
- Define $Z := V(C)$, which is a union of coordinate planes (of codimension at least 2). There is description of Z in terms of so called *primitive collections*

X_Σ as a quotient

- As $G \subseteq (\mathbb{C}^*)^{\Sigma(1)}$, there is a well defined action of G on $\mathbb{C}^{\Sigma(1)} \setminus Z$, and we get

$$X_\Sigma \simeq \mathbb{C}^r \setminus Z//G.$$

- In the case of $\mathbb{P}^{n-1}$, the unique primitive collection is $\{e_0, e_1, \dots, e_n\}$, and we get $Z = \{(0, \dots, 0)\}$.
- $G \simeq \mathbb{Z} \otimes \mathbb{C}^* = \mathbb{C}^*$. As expected, the action of G is the standard action $\lambda \cdot x = (\lambda x_0 : \dots : \lambda x_n)$.

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The End

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Gracias por su atención!

One comprehensive reference:

The bible "Toric varieties" by Cox, Little, Schenck, AMS, 2011, 841 pp.

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